Homomesy in products of three chains and multidimensional recombination

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Outline

- For order ideals of $[a] \times [b]$, Propp and Roby have a homomesy result for the actions of rowmotion and promotion with the cardinality statistic.
- For the homomesy result involving rowmotion, Einstein and Propp used a technique called recombination to produce an alternate, elegant proof.
- I generalize recombination to higher dimensions to show a higher dimensional homomesy result on order ideals of $[2] \times [a] \times [b]$.

Main Topics

1. Toggles, rowmotion, and promotion
2. Homomesy
3. Higher dimensional promotion
4. Recombination
5. Proof sketch of the homomesy result

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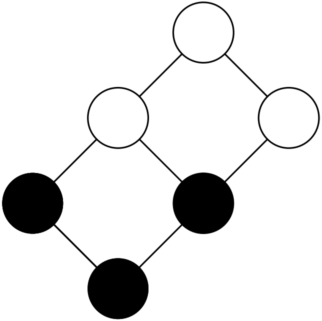
Order ideals

Definition

Let P be a poset. A subset I of P is called an *order ideal* if for any $t \in I$ and $s \leq t$ in P, then $s \in I$.

Example:



What is a toggle?

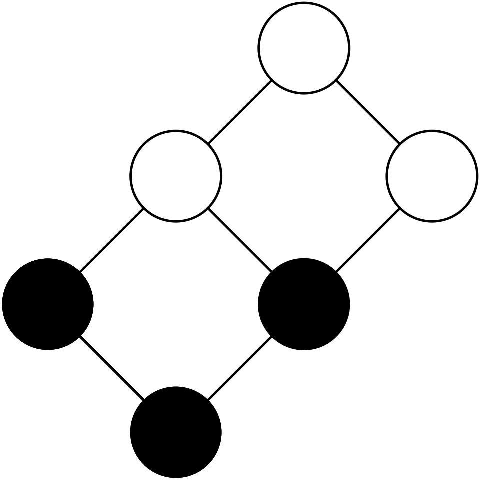
Let P be a poset and $J(P)$ its set of order ideals.

Definition

For any $e \in P$, the *toggle* $t_e : J(P) \to J(P)$ is defined as follows:

$$t_e(I) = \begin{cases} I \cup \{e\} & \text{if } e \notin I \text{ and } I \cup \{e\} \in J(P) \\ I \setminus \{e\} & \text{if } e \in I \text{ and } I \setminus \{e\} \in J(P) \\ I & \text{otherwise} \end{cases}$$

Toggle example



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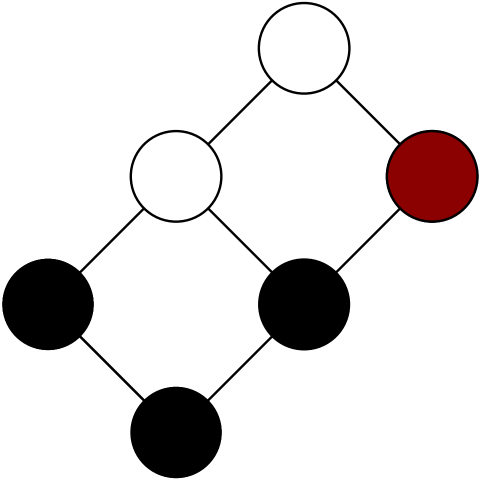

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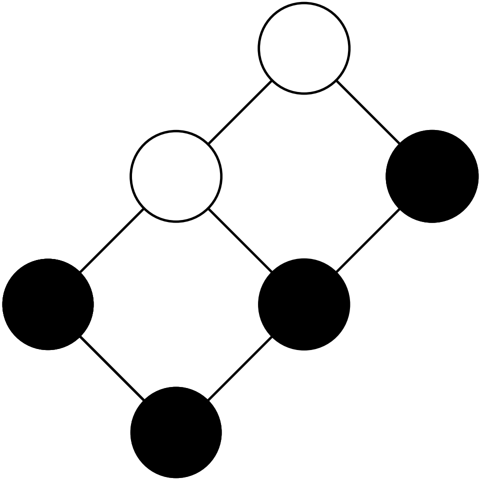

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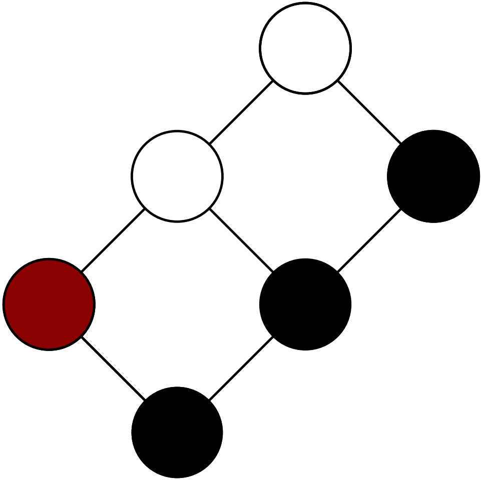

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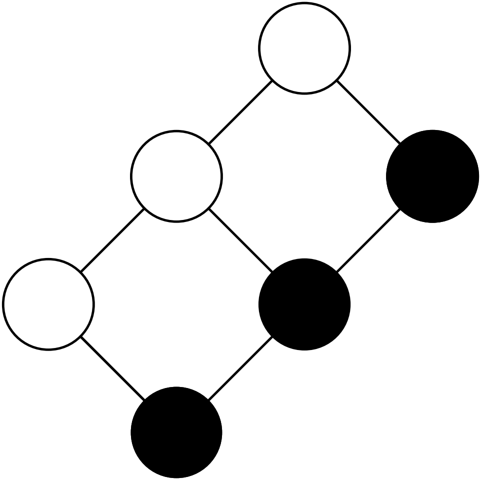

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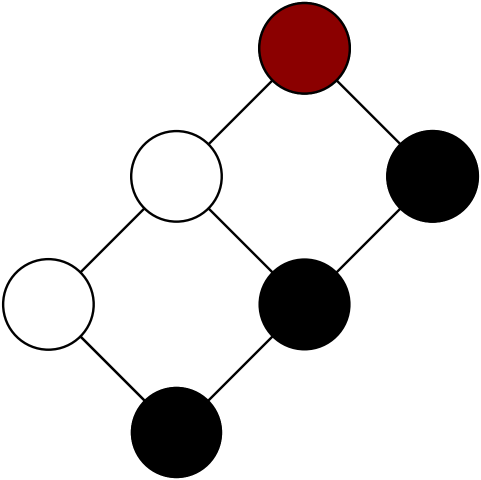

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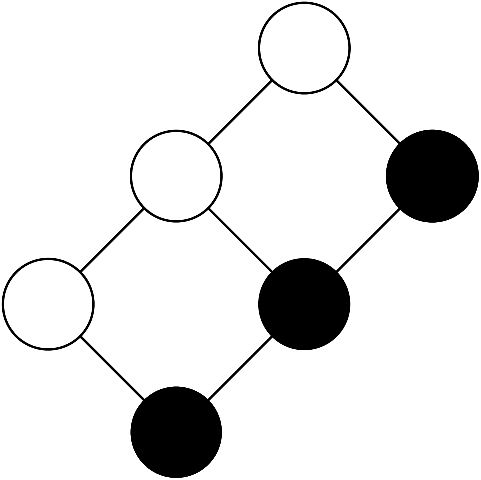


Rowmotion defined

There are two ways we can think of rowmotion.

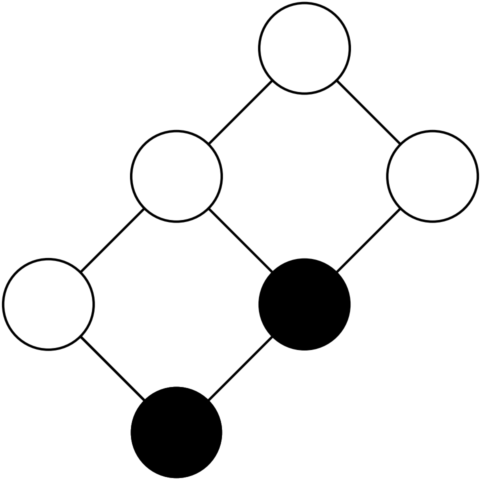
Definition

Let P be a poset and $I \in J(P)$. $\text{Row}(I)$ is the order ideal generated by the minimal elements of $P \setminus I$.

Theorem (Cameron and Fon-der-Flaass)

Rowmotion can be performed by toggling a poset from top to bottom.

Rowmotion example



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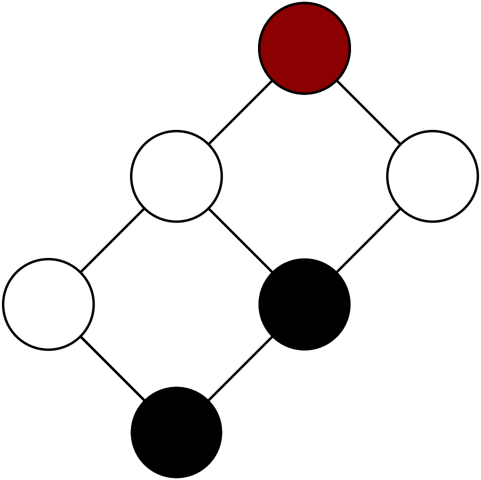

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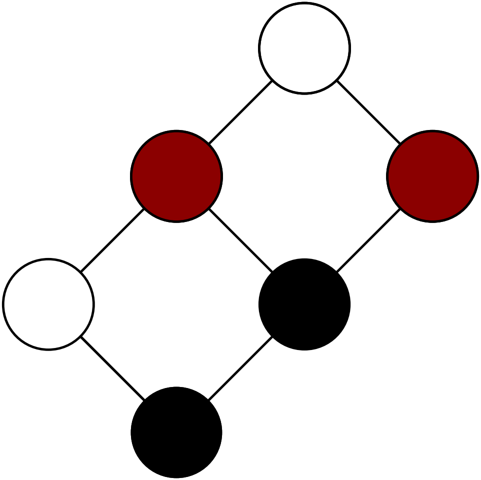

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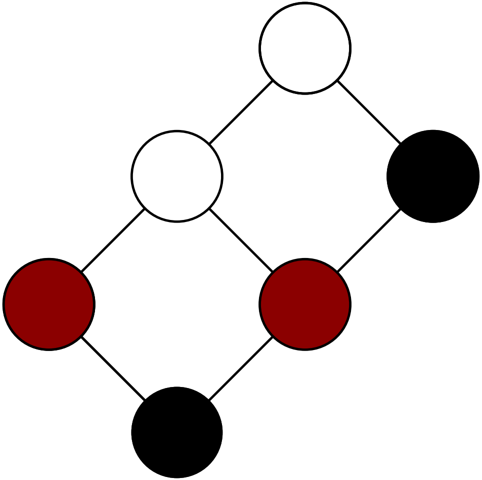

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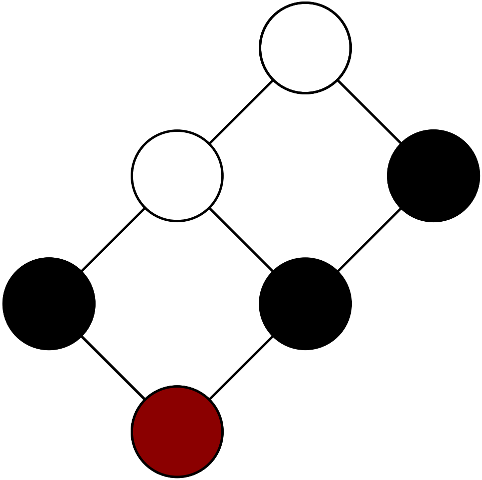

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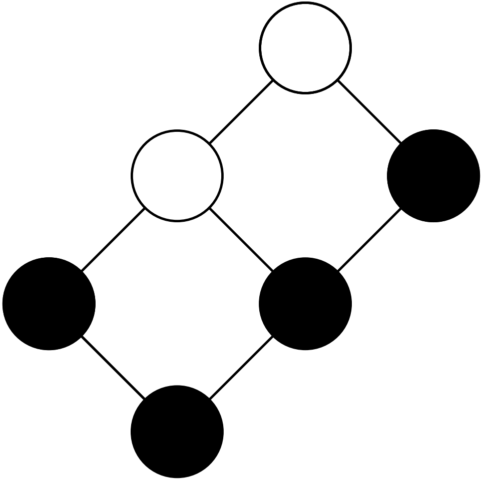

Why use toggles?

- Rowmotion toggles our poset from top to bottom.
- We can define, analogously, *promotion* which toggles our poset from left to right.
- Striker and Williams showed there is an equivariant bijection between orbits under rowmotion and promotion (i.e. they have the same orbit structure).

Main Topics

Homomesy defined

Definition (Propp and Roby)

The triple (S, τ, f) exhibits homomesy if over every orbit of the action $\tau : S \rightarrow S$, the average of the statistic f over the orbit is the same. If such an average c exists, we will say the triple is c-mesic.

The product of two chains

Theorem (Propp and Roby)

Order ideals of $[a] \times [b]$ *under promotion with cardinality statistic are* c*-mesic where* $c = ab/2$.

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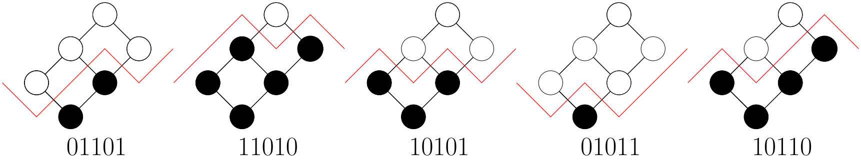
Order ideals of $[a] \times [b]$ *under rowmotion with cardinality statistic are* c*-mesic where* $c = ab/2$.

Einstein and Propp discovered an elegant proof technique for the second result; they named this recombination.

Promotion example



$$\frac{2+5+3+1+4}{5} = 3$$

The previous theorem says on every orbit, the average value of the cardinality statistic is 3.

Does a similar result hold for a higher dimensional product of chains?

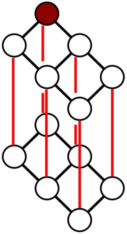
Main Topics

Promotion on a higher dimensional product of chains

Definition (Dilks, Pechenik, Striker)

Let P be a poset with n-dimensional lattice projection and let v be an n-dimensional vector with entries ± 1. Define *promotion with respect to* v, Pro_v, by toggling elements on the hyperplanes $\langle x, v \rangle = i$, sweeping through from largest i to smallest.

Example: $\text{Pro}_{(1,1,1)}$



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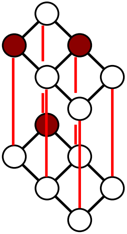

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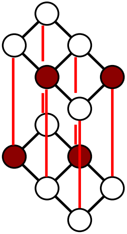

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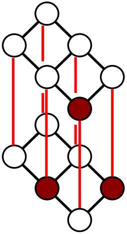

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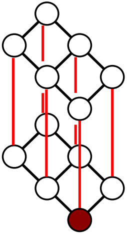

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- $\text{Pro}_{(1,1,1)}$ is Row.
- Dilks, Pechenik, and Striker showed there exists an equivariant bijection between different promotions (i.e. they have the same orbit structure).

Homomesy result

Theorem (V.)

Let $v = (\pm 1, \pm 1, \pm 1)$. Order ideals of $[2] \times [a] \times [b]$ under Pro_v with cardinality statistic are c-mesic with $c = ab$.

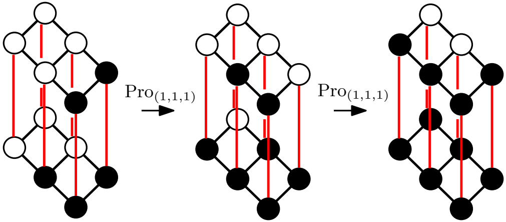
The proof has two main components:

- A generalization of recombination to create a bijection between the orbits of different promotions
- A connection to increasing tableaux to show the theorem for $v = (1, 1, -1)$

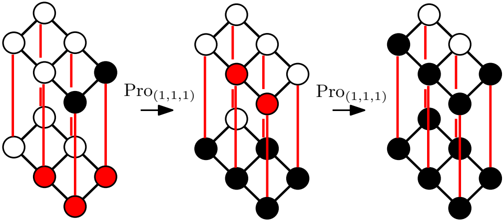
Main Topics

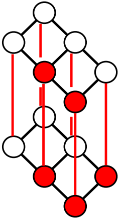
2-dimensional recombination was originally defined by Einstein and Propp.

3-dimensional example:



Recombination





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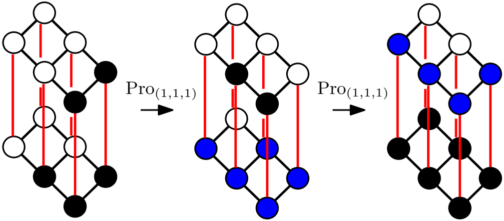

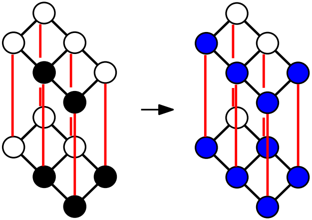

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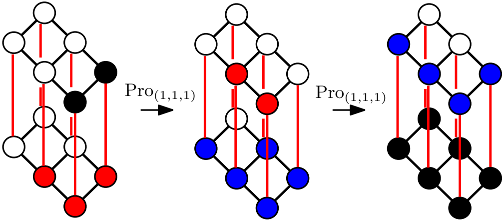

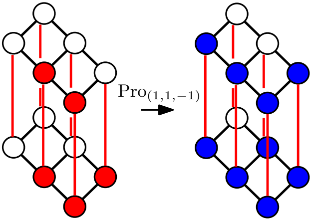

General recombination result

Theorem (V.)

Let P be a poset with n-dimensional lattice projection and $I \in J(P)$. Let v and u be n-dimensional vectors with entries ± 1 such that v and u differ in one component. Then $\mathrm{Pro}_u(\Delta_v^\gamma I) = \Delta_v^\gamma(\mathrm{Pro}_v(I))$.

In other words, recombination works in this setting.

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Proving the homomesy theorem

Theorem (V.)

Let $v = (\pm 1, \pm 1, \pm 1)$. Order ideals of $[2] \times [a] \times [b]$ under Pro_v with cardinality statistic are c-mesic with $c = ab$.

The proof has two main components:

- A generalization of recombination to create a bijection between the orbits of different promotions
- A connection to increasing tableaux to show the theorem for $v = (1, 1, -1)$

Increasing tableaux

Definition

An increasing tableau is a filling of a Young diagram with strictly increasing rows and columns. The set of increasing tableaux of shape λ with largest entry q is denoted $\mathrm{Inc}^q(\lambda)$.

Example:

1	2	4
2	4	5
6		

A useful bijection

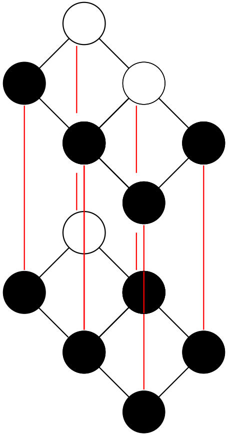
Theorem (Dilks, Pechenik, Striker)

There exists a bijection between $J([a] \times [b] \times [c])$ *and* $\mathrm{Inc}^{a+b+c-1}(a \times b)$.

Corollary

There exists a bijection between $J([2] \times [a] \times [b])$ *and* $\mathrm{Inc}^{a+b+1}(2 \times a)$.

Bijection example



↔

0	1	2
2	2	2

↔

1	3	5
4	5	6

$\text{Pro}_{(1,1,-1)}$ on $J([a] \times [b] \times [c])$ corresponds to Pechenik's K-promotion on increasing tableaux.

A K-promotion result

Theorem (Bloom, Pechenik, Saracino)

Let λ be a $2 \times n$ rectangle for any n and let σ_λ be the statistic of summing the entries in the boxes of λ. Then for any q, $\mathrm{Inc}^q(\lambda)$ under K-Pro with statistic σ_λ is homomesic.

This shows $\mathrm{Pro}_{(1,1,-1)}$ on $J([2] \times [a] \times [b])$ with cardinality statistic is homomesic. Our theorem follows using recombination.

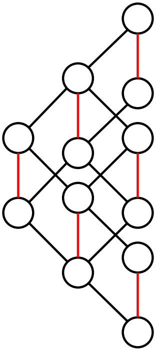
Related Remarks and Corollaries

- Rush and Wang showed order ideals of any minuscule poset exhibit homomesy under rowmotion with cardinality statistic.
- The first of my main results can be rephrased as a 2-chain cross a type A minuscule poset.

Related Remarks and Corollaries

Corollary (V.)

For any $v = (\pm 1, \pm 1, \pm 1)$*, order ideals of a 2-chain cross a type B minuscule poset under* Pro_v *with cardinality statistic are homomesic.*



Let $v = (\pm 1, \pm 1, \pm 1)$.

- Order ideals of $[3] \times [3] \times [3]$ under Pro_v with cardinality statistic are c-mesic with $c = 27/2$.
- Order ideals of $[3] \times [3] \times [4]$ under Pro_v with cardinality statistic are not homomesic.
- Order ideals of $[2] \times [2] \times [2] \times [2]$ under Pro_v with cardinality statistic are c-mesic with $c = 8$.
- Order ideals of $[2] \times [2] \times [2] \times [3]$ under Pro_v with cardinality statistic are not homomesic.
- Order ideals of $[2] \times [2] \times [2] \times [2] \times [2]$ under Pro_v with cardinality statistic are not homomesic.

Related Remarks and Corollaries

- Using Pechenik's homomesy result on the frame of increasing tableaux, we obtain a homomesy result on the "boundary" of $[a] \times [b] \times [c]$.
- We have refined homomesy results on rotationally symmetric "columns" when $P = [2] \times [a] \times [b]$ and when $P = [a] \times [b] \times [c]$ using the "boundary" of the poset.

Thanks!

C. Vorland, Homomesy in products of three chains and multidimensional recombination, http://arxiv.org/abs/1705.02665